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THE BRAID INDEX IS NOT ADDITIVE FOR THE CONNECTED SUM OF 2-KNOTS

SEIICHI KAMADA, SHIN SATOH, AND MANABU TAKABAYASHI

Abstract. Any 2-dimensional knot K can be presented in a braid form, and its braid index, $\mathrm{Braid}(K)$, is defined. For the connected sum $K_1 \# K_2$ of 2-knots K_1 and K_2, it is easily seen that $\mathrm{Braid}(K_1 \# K_2) \leq \mathrm{B}(K_1) + \mathrm{B}(K_2) - 1$ holds. Birman and Menasco proved that the braid index (minus one) is additive for the connected sum of 1-dimensional knots; the equality holds for 1-knots. We prove that the equality does not hold for 2-knots unless K_1 or K_2 is a trivial 2-knot. We also prove that the 2-knot obtained from a granny knot by Artin's spinning is of braid index 4, and there are infinitely many 2-knots of braid index 4.

1. Introduction

By the Alexander theorem [1], any 1-dimensional knot k can be presented in a braid form. The *braid index* of k is the minimum number among the degrees of all closed braids that are equivalent to k, which is denoted by $\mathrm{Braid}(k)$. When 1-knots k_1 and k_2 are presented by an m_1-braid b_1 and an m_2-braid b_2 respectively, one can construct an $(m_1 + m_2 - 1)$-braid $b_1 \# b_2$ which presents the connected sum $k_1 \# k_2$. Thus there is an obvious inequality

$$\mathrm{Braid}(k_1 \# k_2) \leq \mathrm{Braid}(k_1) + \mathrm{Braid}(k_2) - 1. \tag{1}$$

Actually, J. S. Birman and W. Menasco [4] proved that the braid index (minus one) is additive under the connected sum.

Theorem 1 (Birman-Menasco [4])**.** *For the connected sum $k_1 \# k_2$ of 1-knots k_1 and k_2,*

$$\mathrm{Braid}(k_1 \# k_2) = \mathrm{Braid}(k_1) + \mathrm{Braid}(k_2) - 1. \tag{2}$$

In this paper we consider an analogous problem for 2-knots. The notion of a 2-dimensional braid was introduced by O. Ya. Viro [29], and it is proved in [12] that any 2-knot (or any closed oriented 2-submanifold of $\mathbb{R}^4$) can be presented in a braid form. A similar notion, called a braided surface, was studied by L. Rudolph [25, 26, 27]. J. S. Carter and M. Saito have been studying the braid presentations of 2-knots, [5, 6, 7, 8]. Refer to [10, 11, 12, 13, 16, 18, 19] for the first author's research in this field. Another kind of braid presentation of a 2-knot was studied by F. González-Acuña [9]. 2-dimensional braids are also related to the braid monodromies of stable branch curves in complex geometry [22, 23].

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The *braid index*, which is denoted by $\mathrm{Braid}(K)$, of a 2-knot K is defined to be the minimum number among the degrees of all simple closed 2-dimensional braids that are equivalent to K. The braid index is 1 if and only if K is a trivial 2-knot. There exist no 2-knots of braid index 2 (cf. [10]). It is proved in [10] that if the braid index of a 2-knot K is 3, then K is a ribbon 2-knot, that is, it can be obtained from some mutually disjoint trivial 2-spheres in $\mathbb{R}^4$ by surgery along some 1-handles; cf. [20]. The converse is not true; for example, the 2-knot obtained from a figure eight knot by Artin's spinning construction [2] is ribbon, and the braid index is not 3, [16].

For a 2-dimensional m_1-braid S_1 and a 2-dimensional m_2-braid S_2, we can construct a 2-dimensional $(m_1 + m_2 - 1)$-braid $S_1 \# S_2$ such that the closure in $\mathbb{R}^4$ is equivalent to the connected sum of the closures of S_1 and S_2. The following is just an analogy of the inequality (1).

Lemma 2 ([17]). *For 2-knots K_1 and K_2,*

$$\mathrm{Braid}(K_1 \# K_2) \leq \mathrm{Braid}(K_1) + \mathrm{Braid}(K_2) - 1, \tag{3}$$

where $K_1 \# K_2$ is the connected sum of K_1 and K_2.

It is obvious that

$$\mathrm{Braid}(K_1 \# K_2) = \mathrm{Braid}(K_1) + \mathrm{Braid}(K_2) - 1, \tag{4}$$

provided that K_1 or K_2 is a trivial 2-knot. Surprisingly, this is the only case where the equality of (2) holds for the connected sum of 2-knots. The following theorem is our main result.

Theorem 3. *If neither K_1 nor K_2 is a trivial 2-knot, then*

$$\mathrm{Braid}(K_1 \# K_2) < \mathrm{Braid}(K_1) + \mathrm{Braid}(K_2) - 1. \tag{5}$$

There exist infinitely many ribbon 2-knots of braid index 3 (cf. Lemma 15 or [16]). Using Theorem 3, we prove the following.

Corollary 4. *There exist an infinite series of ribbon 2-knots of braid index 4 which includes the 2-knot obtained from a granny knot by Artin's spinning.*

This paper is organized as follows. In Sections 2 and 3, we review the notions of a 2-dimensional braid and the chart presentation. In Sections 4 and 5, we prove Theorem 3 by using the chart presentation. A proof of Corollary 4 is given in Section 6.

2. Simple 2-dimensional braids

Let D_1 and D_2 be 2-disks and X_m a fixed set of m distinct interior points of D_1, where m is a positive integer. Let $\pi : D_1 \times D_2 \to D_2$ be the projection map. A 2-*dimensional m-braid* (or a 2-*dimensional braid of degree m*) is a compact oriented surface S properly embedded in $D_1 \times D_2$ such that

(i) the restriction map $\pi|_S : S \to D_2$ of the projection π to S is an m-fold branched covering map of D_2, and
(ii) the boundary ∂S is the trivial closed m-braid $X_m \times \partial D_2$ in the solid torus $D_1 \times \partial D_2$.

Moreover, if an additional condition that

(iii) the branched covering $\pi|_S : S \to D_2$ is simple, that is, $\#\Big(S \cap \pi^{-1}(x)\Big) = m-1$ or m for each $x \in D_2$,

is satisfied, then the 2-dimensional m-braid S is said to be *simple*.

Two simple 2-dimensional braids S and S' are *equivalent* if they are ambiently isotopic by an ambient isotopy $\{h_t\}_{t\in[0,1]}$ of $D_1 \times D_2$ such that for each $t \in [0,1]$, $h_t(S)$ is a simple 2-dimensional braid. This definition is different from that in [10], however these definitions are equivalent [14]. Moreover, it is proved in [14] that simple 2-dimensional m-braids S and S' are equivalent if and only if there exists a one-parameter family $\{S_t\}_{t\in[0,1]}$ of simple 2-dimensional m-braids with $S_0 = S$ and $S_1 = S'$. We usually regard equivalent 2-dimensional braids as the same.

Divide the 2-disk D_2 into two 2-disks $D_2^{(1)}$ and $D_2^{(2)}$ by a properly embedded arc in D_2. For simple 2-dimensional m-braids S_1 and S_2, we define the product $S_1 \cdot S_2$ to be the simple 2-dimensional m-braid such that the restriction to $D_1 \times D_2^{(i)}$ is a copy of S_i for each $i \in \{1,2\}$. This product is uniquely determined by S_1 and S_2 up to equivalence.

Let S be a simple 2-dimensional m-braid. Identify D_2 with the product $I_1 \times I_2$ of the unit intervals I_1 and I_2. For each $t \in I_2 = [0,1]$, we denote the cylinder $D_1 \times I_1 \times \{t\} \subset D_1 \times D_2$ by $(D \times I)_t$ and put $b_t = S \cap (D \times I)_t$. Then b_t is a 1-dimensional m-braid in $(D\times I)_t$ for all t but a finite number of exceptional values. For each exceptional value t, the singular braid b_t has double points in its strands; see Figure 1. We call the one-parameter family $\{b_t\}_{t\in[0,1]}$ a *motion picture* of S. Note that both b_0 and b_1 are trivial m-braids, since $\partial S = X_m \times \partial D_2 \subset D_1 \times D_2$.

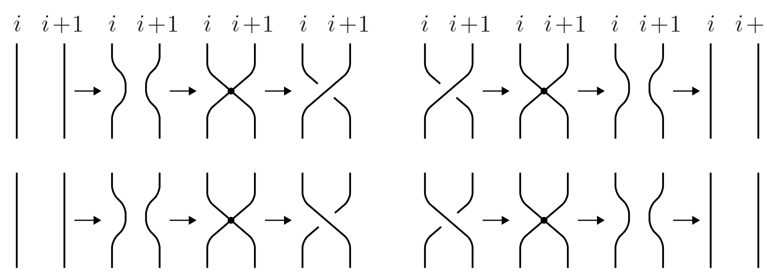

FIGURE 1.

When $\{b_t^{(1)}\}_{t\in[0,1]}$ and $\{b_t^{(2)}\}_{t\in[0,1]}$ are motion pictures of S_1 and S_2 respectively, we have a motion picture of the product $S_1 \cdot S_2$ of S_1 and S_2 by taking the usual product $b_t^{(1)} \cdot b_t^{(2)}$ as braids for each $t \in [0,1]$.

Embed the 4-disk $D_1 \times D_2$ in $\mathbb{R}^4$. Let S be a 2-dimensional m-braid. The boundary of S is a trivial link in the 3-sphere $\partial(D_1 \times D_2)$. Attaching m sheets of 2-disks along the boundary of S in $\mathbb{R}^4 \setminus \mathrm{int}(D_1 \times D_2)$ in the obvious way, we have a closed oriented surface embedded in $\mathbb{R}^4$. It is called the *closure* of S in $\mathbb{R}^4$ and denoted by $\widehat{S}$.

When S is described by a motion picture $\{b_t\}_{t\in[0,1]}$, we have a motion picture of the closure $\widehat{S}$ as follows: For each $t \in [0,1]$, assume that b_t is lying in the hyperplane $\mathbb{R}^3 \times \{t\} \subset \mathbb{R}^3 \times \mathbb{R}^1 = \mathbb{R}^4$ and consider the closure $\widehat{b}_t$ of b_t in the hyperplane. The trace of the closed braids forms a properly embedded surface in $\mathbb{R}^3 \times [0,1]$. Since

b_0 and b_1 are trivial m-braids, $\widehat{b}_0$ and $\widehat{b}_1$ are trivial links in $\mathbb{R}^3 \times \{0\}$ and $\mathbb{R}^3 \times \{1\}$, respectively. Attaching m trivial disks in the lower side and m trivial disks in the upper side, we have a closed surface in $\mathbb{R}^4$, which is the closure of S.

Theorem 5 (Viro [29], [12]). *Any 2-knot (or closed oriented 2-submanifold of $\mathbb{R}^4$) is equivalent to the closure of a simple 2-dimensional m-braid for some m.*

Definition 6. The *braid index* of a 2-knot K, denoted by Braid(K), is the minimum number among the degrees of all simple 2-dimensional braids whose closures are equivalent to K.

The following theorem corresponds to the Markov theorem for 1-knots [3, 21].

Theorem 7 ([13, 18, 19]). *Let S and S' be simple 2-dimensional braids. The closure of S is equivalent to that of S' if and only if there exists a finite sequence*

$$S = S_0, S_1, \ldots, S_n = S'$$

of simple 2-dimensional braids such that each S_i, $i = 1, 2, \ldots, n$, is obtained from S_{i-1} by a braid ambient isotopy, a conjugation move, a stabilization move or a destabilization move.

Here, two simple 2-dimensional m-braids S and S' are *braid ambient isotopic* if they are ambiently isotopic by an ambient isotopy $\{h_t\}_{t\in[0,1]}$ of $D_1 \times D_2$ such that for each $t \in [0, 1]$, $h_t(S)$ is a (possibly non-simple) 2-dimensional m-braid. If S and S' are equivalent, then they are braid ambient isotopic. It is unknown whether the converse holds or not. Conjugation, stabilization and destabilization moves are explained in the next section in terms of the chart presentation.

3. The chart presentation

In this section, we recall the chart presentation of a simple 2-dimensional braid (cf. [8, 10, 19]). An *m-chart* is a finite graph in the interior of a 2-disk D_2, which may be empty or have *hoops* (that are closed edges without vertices), satisfying the following conditions:

(i) Every vertex has valency 1, 4, or 6.
(ii) Every edge is oriented and labeled by an integer from $\{1, 2, \ldots, m-1\}$.
(iii) Around each 4-valent vertex, diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i-j| > 1$.
(iv) Around each 6-valent vertex, three consecutive edges are oriented inward and the others are oriented outward. These six edges are labeled i and $i+1$ alternately around the vertex for some i.

See Figure 2. We do not distinguish two m-charts if they are ambiently isotopic in D_2. A *free edge* means a single edge whose endpoints are 1-valent vertices.

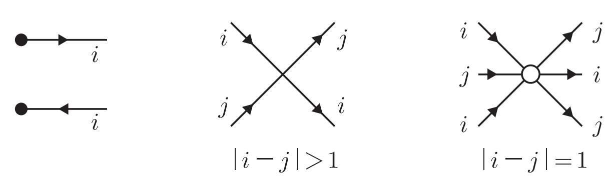

Figure 2.

Divide D_2 into two 2-disks $D_2^{(1)}$ and $D_2^{(2)}$ by a properly embedded arc in D_2 as in Section 2. For two charts Γ_1 and Γ_2 in D_2, we denote by $\Gamma_1 \cdot \Gamma_2$ the chart that is the union of a copy of Γ_1 in $D_2^{(1)}$ and a copy of Γ_2 in $D_2^{(2)}$.

Let Γ be an m-chart in $D_2 = I_1 \times I_2$. Put $\ell_t = I_1 \times \{t\}$ for each $t \in I_2 = [0,1]$. Modifying Γ by an isotopy of D_2, we assume that for all t but a finite number of exceptional values, there are no vertices of Γ on ℓ_t and the intersection of ℓ_t and Γ consists of transverse double points. Also, for each exceptional value t, the intersection of ℓ_t and Γ consists of transverse double points and a single point that is a vertex of Γ or a maximal or minimal point of an edge of Γ.

For a regular value t, assign each point of the intersection $\ell_t \cap \Gamma$ a letter σ_i (or σ_i^{-1}) if its intersecting edge of Γ is labeled i and oriented from left to right (or right to left). We read all these letters along ℓ_t to obtain a word on the standard generators $\sigma_1, \dots, \sigma_{m-1}$ of the m-braid group. We denote this braid word by $w_\Gamma(\ell_t)$ and call it the *intersection braid word* along ℓ_t with respect to the chart Γ. It is proved in [10] that there is a simple 2-dimensional m-braid S_Γ whose motion picture $\{b_t\}_{t\in[0,1]}$ satisfies the condition that for each regular value t, the m-braid $b_t = S_\Gamma \cap (D \times I)_t$ is presented by the braid word $w_\Gamma(\ell_t)$. It is also proved that the equivalence class of S_Γ is uniquely determined from (the ambient isotopy class of) the chart Γ. Conversely, any simple 2-dimensional m-braid S is equivalent to S_Γ for some (not unique) m-chart Γ. In this situation, we say that Γ is a *chart presentation* of S, or S is presented by the chart Γ. Refer to [8, 10, 19] for more details.

Example 8. The top of Figure 3 shows an example of a 4-chart, and the 2-dimensional braid presented by this chart is depicted in the middle and bottom of the figure by the motion picture, where $t_1, \dots, t_4$ are exceptional values.

Local operations on m-charts listed below are called CI-, CII-, and CIII-*moves*, respectively. A C-*move* is one of them or its inverse. Two m-charts are C-*move equivalent* if they are related by a sequence of C-moves (up to ambient isotopy in D_2). Let Γ be an m-chart in D_2.

(CI) For a 2-disk E in D_2 such that $\Gamma \cap E$ has no 1-valent vertices, change $\Gamma \cap E$ arbitrarily as long as it has no 1-valent vertices.

(CII) Suppose that an edge e connects a 4-valent vertex v^4 and a 1-valent vertex v^1. Remove e and v^4, attach v^1 to the edge of v^4 opposite to e, and connect the other edges in a natural way (see Figure 4).

(CIII) Let a 1-valent vertex v^1 and a 6-valent vertex v^6 be connected by an edge e. Suppose that e is not the middle edge of a set of three consecutive edges attaching to v^6 which are oriented in the same direction. Then, remove e and v^6, attach v^1 to the edge of v^6 opposite to e, and connect the other edges in a natural way (see Figure 4).

Figure 5 shows two examples of CI-moves. The first one in the figure is called a *channel change move*.

Theorem 9 ([10, 15])**.** *Two m-charts Γ and Γ' are C-move equivalent if and only if the simple 2-dimensional m-braids presented by Γ and Γ' are equivalent.*

We consider two more operations on charts. Let Γ be an m-chart in D_2.

(MI) Add some hoops (simple loops) parallel to ∂D_2 surrounding Γ. Each of the hoops is oriented and labeled by an integer from $\{1, \dots, m-1\}$ arbitrarily.

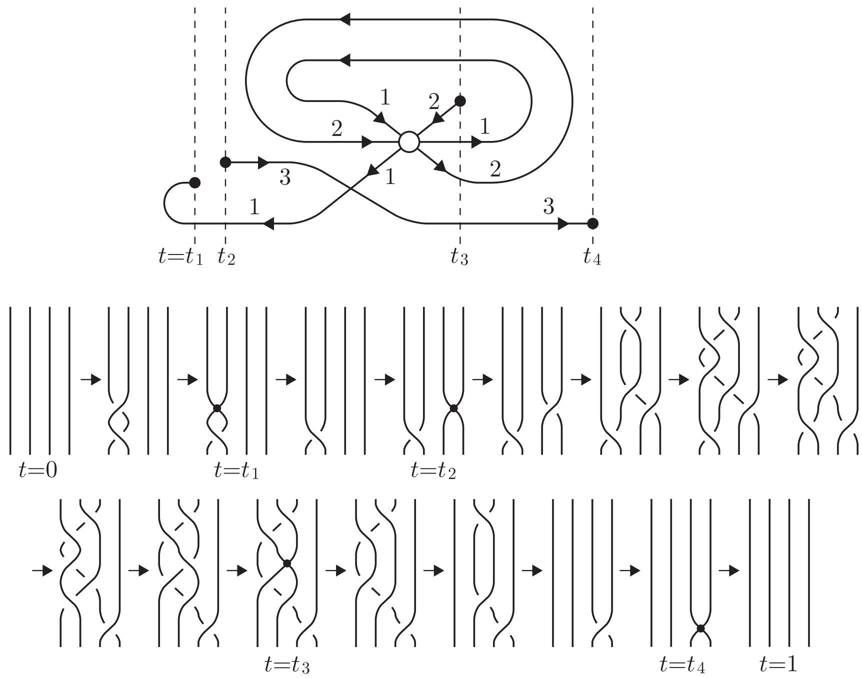

Figure 3.

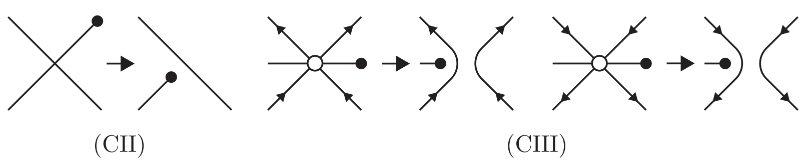

Figure 4.

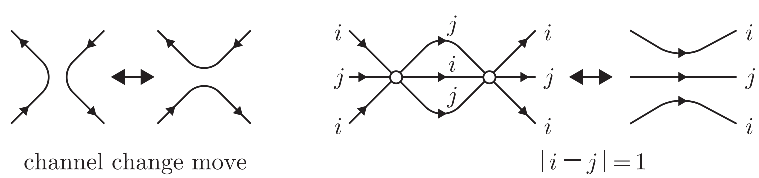

Figure 5.

(MII) Insert a free edge labeled m near ∂D_2, and regard the union of Γ and the free edge as an $(m+1)$-chart.

Let Γ be a chart presentation of a simple 2-dimensional m-braid S. We say that a simple 2-dimensional m-braid S' is obtained from S by a *conjugation move* if S' has a chart presentation that is obtained from Γ by an MI-move or its inverse. A simple 2-dimensional $(m+1)$-braid S' is obtained from S by a *stabilization move* if S' has a chart presentation that is obtained from Γ by an MII-move. A *destabilization move* is the inverse operation of a stabilization move. See Figures 6 and 7 for the motion pictures. These definitions stated in terms of the chart presentation [17] are compatible with the original definitions in [13] stated geometrically, and the definitions in [18] stated in terms of braid systems. Refer to [19] for details and for a complete proof of Theorem 7. Therefore, if two charts are related by a sequence of C-moves, MI-moves and MII-moves, then the closures of their presenting 2-dimensional braids are equivalent.

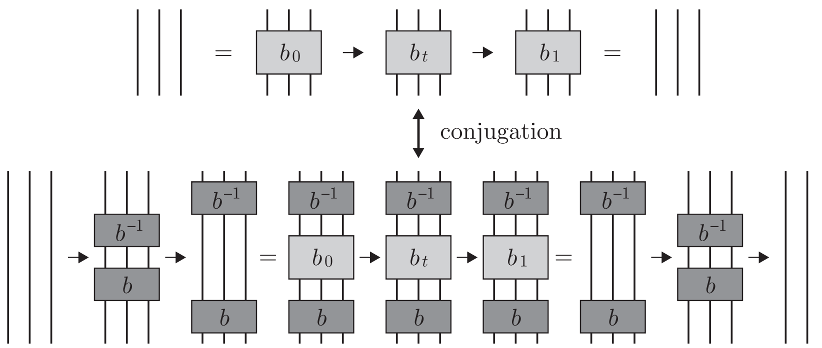

FIGURE 6.

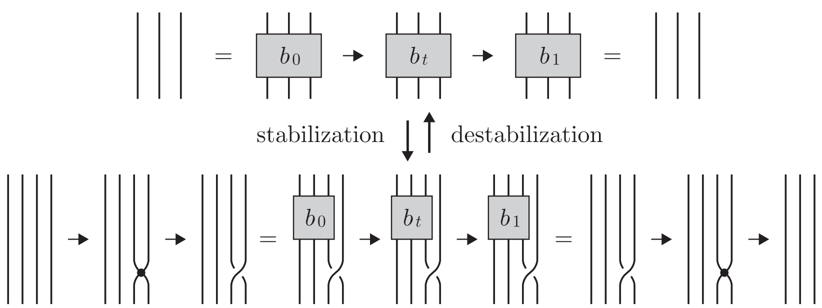

FIGURE 7.

4. THE COMPOSITE OF 2-DIMENSIONAL BRAIDS

Let p and q be non-negative integers. For a (1-dimensional) m-braid b, we denote by $\iota_p^q(b)$ the $(p+m+q)$-braid that is obtained from b by introducing p trivial strands on the left side of b and q trivial strands on the right side. In other words, when a braid b is expressed by a word on the standard generators $\sigma_1, \ldots, \sigma_{m-1}$, the $(p+m+q)$-braid $\iota_p^q(b)$ is obtained by shifting the subscripts i of σ_i to $i+p$ ($i = 1, \ldots, m-1$), and considering these to be generators in the braid group B_{p+m+q}.

The *composite* or the *connected sum* of an m_1-braid b_1 and an m_2-braid b_2, denoted by $b_1 \# b_2$, is the (m_1+m_2-1)-braid

$$\iota_0^{m_2-1}(b_1) \cdot \iota_{m_1-1}^0(b_2)$$

(see Figure 8). The closure of $b_1 \# b_2$ is a connected sum of the closures of b_1 and b_2. The dotted circle in the figure indicates a decomposing 2-sphere in $\mathbb{R}^3$ for the connected sum; namely, it shows the equatorial cross-section of the decomposing 2–sphere with respect to the plane in $\mathbb{R}^3$ on which the knot diagram is constructed.

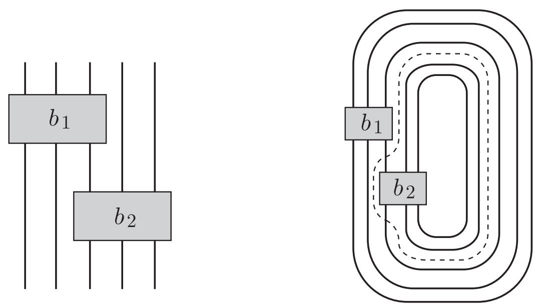

FIGURE 8.

The notion of the composite of 1-dimensional braids is generalized to that of 2-dimensional braids as follows: For a 2-dimensional m-braid S, we denote by $\iota_p^q(S)$ the 2-dimensional $(p+m+q)$-braid that is obtained from S by introducing p trivial 2-disks on the left side of S and q trivial 2-disks on the right side. When S is described by a motion picture $\{b_t\}_{t\in[0,1]}$, then the 2-dimensional $(p+m+q)$-braid $\iota_p^q(S)$ is described by the motion picture $\{\iota_p^q(b_t)\}_{t\in[0,1]}$. Note that we use the same notation ι_p^q for 1- and 2-dimensional braids, but it will cause no confusion.

The *composite* or the *connected sum* of a 2-dimensional m_1-braid S_1 and a 2-dimensional m_2-braid S_2, denoted by $S_1\#S_2$, is the 2-dimensional (m_1+m_2-1)-braid

$$\iota_0^{m_2-1}(S_1)\cdot\iota_{m_1-1}^0(S_2).$$

In other words, a motion picture of the composite $S_1\#S_2$ is obtained from the motion pictures $\{b_t^{(1)}\}_{t\in[0,1]}$ and $\{b_t^{(2)}\}_{t\in[0,1]}$ of S_1 and S_2 by taking the composite braid $b_t^{(1)}\#b_t^{(2)}$ for each $t\in[0,1]$.

Lemma 10 ([17])**.** *The closure of the composite $S_1\#S_2$ is a connected sum of the closures of S_1 and S_2.*

Proof. The trace of the decomposing 2-spheres for the closures of $b_t^{(1)}\#b_t^{(2)}$ in $\mathbb{R}^3\times\{t\}\subset\mathbb{R}^4$ for all $t\in[0,1]$ (as in Figure 8) forms an embedded 3-manifold in $\mathbb{R}^3\times[0,1]\subset\mathbb{R}^4$ which is homeomorphic to $S^2\times[0,1]$. Attaching a 3-ball in the upper side and a 3-ball in the lower side in an obvious way, we have a decomposing 3-sphere in $\mathbb{R}^4$ for a connected sum of the closures of S_1 and S_2. □

Lemma 2 follows from this lemma immediately.

In terms of the chart presentation, the composite of 2-dimensional braids is stated as follows: For an m-chart Γ and non-negative integers p and q, let $\iota_p^q(\Gamma)$ be the $(p+m+q)$-chart obtained from Γ by adding p to the label of every edge of Γ and regarding the chart as an $(p+m+q)$-chart. This notion, $\iota_p^q(\Gamma)$, is compatible with the notion of $\iota_p^q(S)$ for a 2-dimensional braid S: If a 2-dimensional m-braid S has a chart presentation Γ, then the chart $\iota_p^q(\Gamma)$ is a chart presentation of the 2-dimensional $(p+m+q)$-braid $\iota_p^q(S)$. For an m_1-chart Γ_1 and an m_2-chart Γ_2, we denote by $\Gamma_1\#\Gamma_2$ the (m_1+m_2-1)-chart

$$\iota_0^{m_2-1}(\Gamma_1)\cdot\iota_{m_1-1}^0(\Gamma_2).$$

Then $\Gamma_1 \# \Gamma_2$ is a chart presentation of $S_1 \# S_2$, where S_1 and S_2 are 2-dimensional braids presented by Γ_1 and Γ_2.

The following lemma is equivalent to Lemma 10.

Lemma 11 ([17])**.** *Let K_i be a closed oriented surface in $\mathbb{R}^4$ presented by an m_i-chart Γ_i $(i = 1, 2)$. Then the $(m_1 + m_2 - 1)$-chart $\Gamma_1 \# \Gamma_2$ presents a connected sum $K_1 \# K_2$.*

5. Proof of Theorem 3

Let v be a 1-valent vertex of a chart Γ in D_2. The *label* of v stands for the label of the edge adjacent to v. The *sign* of v is $+1$ (or -1) if the adjacent edge is oriented outward from v (or toward v). We say that v is *outermost* if we can connect the vertex v and a point of ∂D_2 by a simple arc α such that $\alpha \cap \Gamma = \{v\}$.

Lemma 12. *Let Γ be an m-chart with 1-valent vertices, where $m \geq 2$. For any $\varepsilon \in \{+1, -1\}$ and $k \in \{1, \dots, m-1\}$, one can transform Γ by C-moves and MI-moves such that there exists an outermost 1-valent vertex with sign ε and label k.*

Proof. Note that half of the 1-valent vertices have positive signs and the other half have negative signs; cf. [10]. Choose a 1-valent vertex whose sign is ε, say v. The composition of a CI-move and a CIII-move illustrated in Figure 9 changes the label i of v into an integer j with $|i-j| = 1$ and $1 \leq j \leq m-1$. Applying this inductively, we may assume that the label of v is the given integer k. Using an MI-move and a CI-move, we can make the vertex v to be outermost as follows: Consider a simple arc α from v to a point of ∂D_2 intersecting Γ transversely, and read the intersection braid word $w_\Gamma(\alpha)$ along α (ignoring the starting point v) as in Section 3. Add some hoops parallel to ∂D_2 oriented and labeled such that the intersection braid word of these hoops along α is the inverse of $w_\Gamma(\alpha)$. This is an MI-move. Applying channel change moves along α as in Figure 10, which is a CI-move, we can remove the intersection of the chart and the arc α, except for the vertex v. □

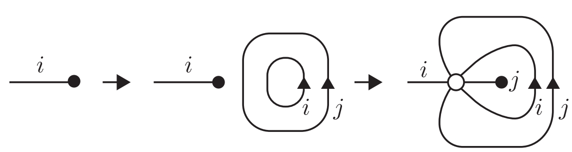

Figure 9.

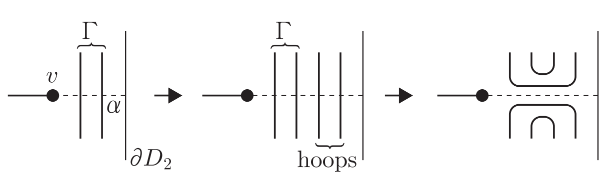

Figure 10.

Let Γ and E be an m-chart and a 2-disk in D_2 respectively such that $\Gamma \cap \partial E$ is empty or consists of transverse double points missing the vertices of Γ. Then we call the intersection $\Gamma \cap E$ a *subchart* of Γ.

Lemma 13. *Let m and ℓ be integers with $1 < m < \ell$. Let Γ be an ℓ-chart which has a subchart Λ whose edges are labeled by integers in $\{m, m+1, \ldots, \ell-1\}$ as shown in the left of Figure 11. Let Γ' be the ℓ-chart obtained from Γ by the replacement illustrated in Figure 11, where Λ' is the subchart obtained from Λ by shifting all of the labels by -1. Then Γ and Γ' present the same 2-dimensional braid.*

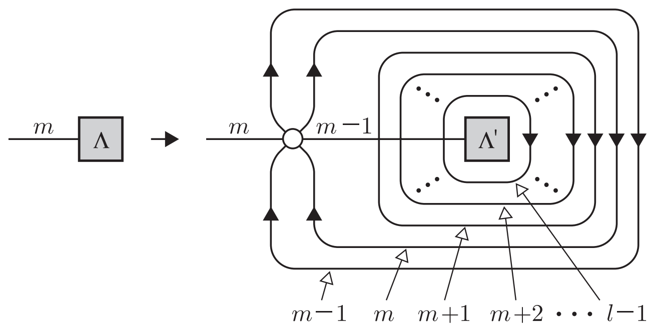

FIGURE 11.

Lemma 14. *Let m and ℓ be integers with $1 < m < \ell$. Let Γ be an ℓ-chart which has a subchart as shown on the left side of Figure 12, and let Γ' be the ℓ-chart obtained from Γ by the replacement illustrated in Figure 12. Then Γ and Γ' present the same 2-dimensional braid.*

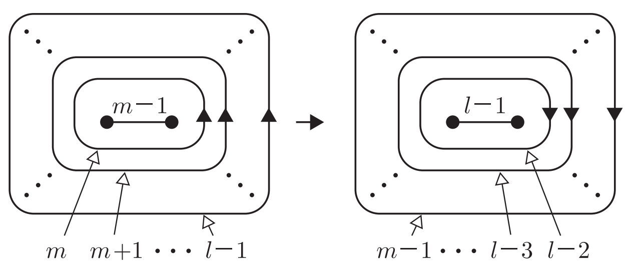

FIGURE 12.

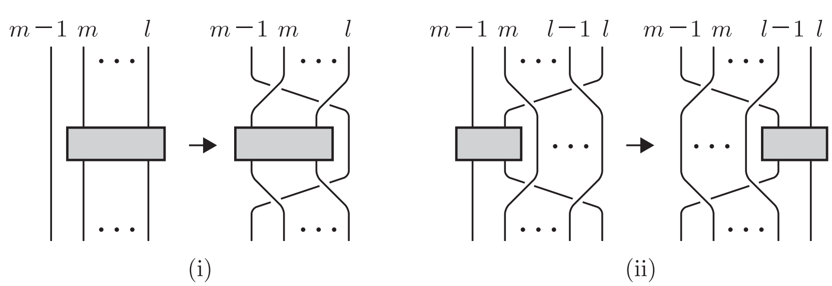

FIGURE 13.

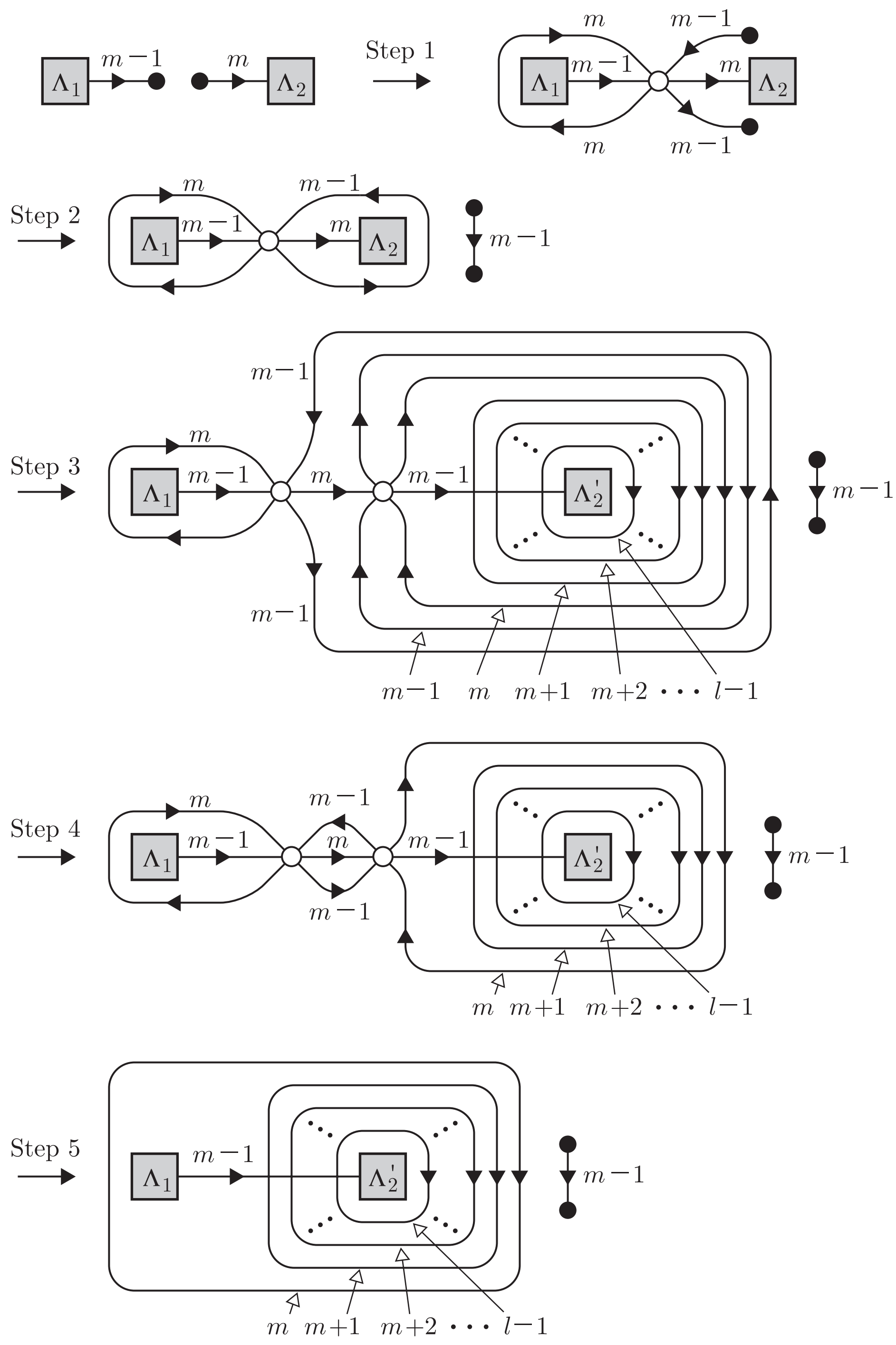
Λ_1
$m-1$
m
Λ_2
Step 1
Λ_2'
Step 2
Step 3
$m+1$
$m+2$
$l-1$
Step 4
Step 5

FIGURE 14.

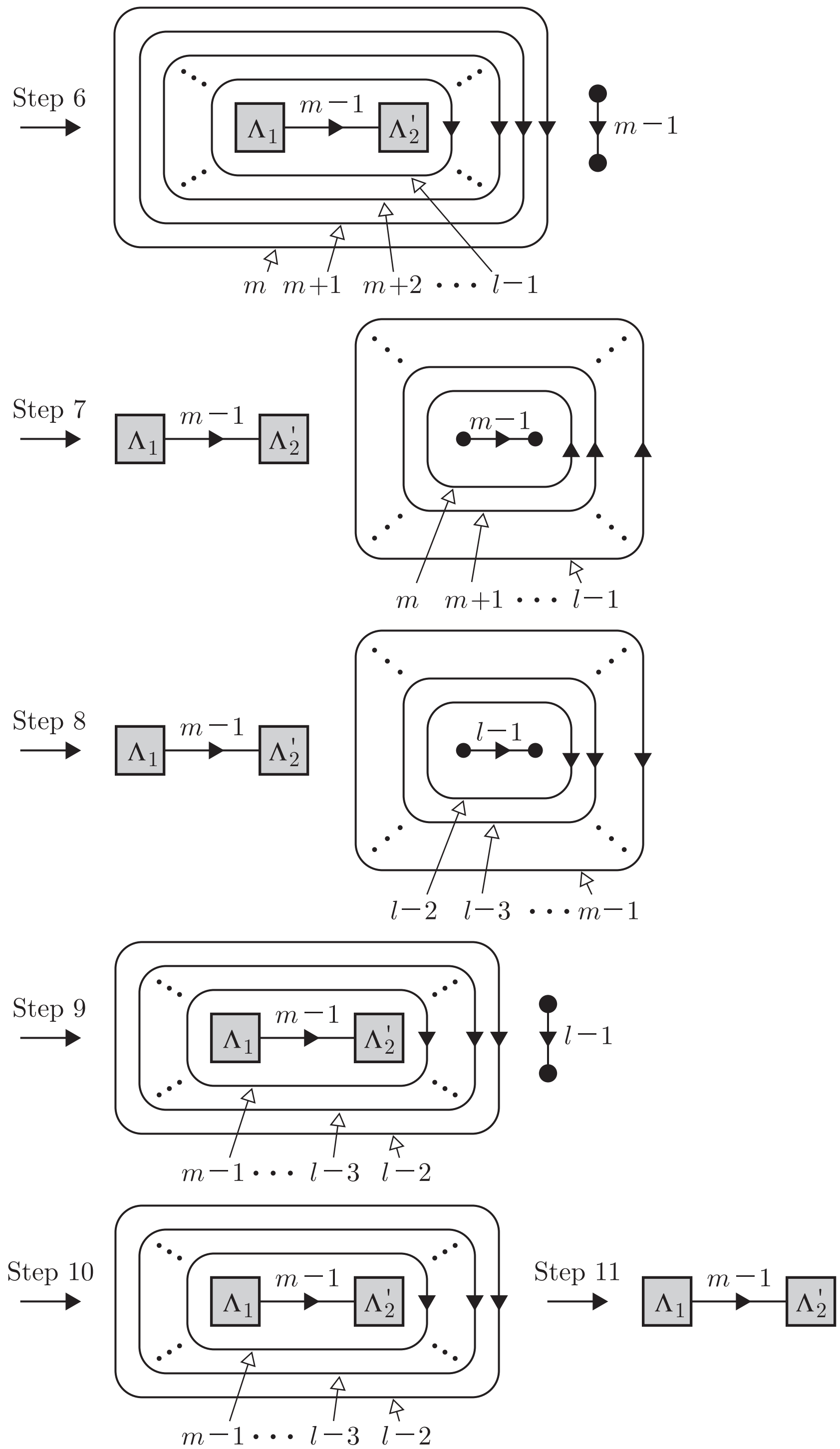
Step 6
Λ_1
$m-1$
Λ_2'
$m-1$
m $m+1$ $m+2$ $\cdots$ $l-1$
Step 7
Λ_1
$m-1$
Λ_2'
$m-1$
m $m+1$ $\cdots$ $l-1$
Step 8
Λ_1
$m-1$
Λ_2'
$l-1$
$l-2$ $l-3$ $\cdots$ $m-1$
Step 9
Λ_1
$m-1$
Λ_2'
$l-1$
$m-1$ $\cdots$ $l-3$ $l-2$
Step 10
Λ_1
$m-1$
Λ_2'
$m-1$ $\cdots$ $l-3$ $l-2$
Step 11
Λ_1
$m-1$
Λ_2'

FIGURE 15.

Proofs of Lemmas 13 *and* 14. Since the braid monodromies of Γ and Γ' are the same, they present the same (up to equivalence) 2-dimensional braid (see [14, 15, 19] for the definition of the braid monodromy and a proof of the fact that the braid monodromy determines the 2-dimensional braid). □

The ideas of Lemmas 13 and 14 are schematized as in Figure 13(i) and (ii), respectively.

Proof of Theorem 3. Let K_i be a non-trivial 2-knot and Γ_i an m_i-chart presenting K_i $(i = 1, 2)$. It is sufficient to prove that the connected sum $K_1\#K_2$ has a chart presentation by an (m_1+m_2-2)-chart. Note that both of Γ_1 and Γ_2 have 1-valent vertices. (If Γ_i has no 1-valent vertices, then the 2-dimensional braid presented by Γ_i is trivial [10], and K_i is the union of m_i trivial 2-spheres in $\mathbb{R}^4$. Since K_i is a non-trivial 2-knot, this case does not occur.) Using Lemma 12, we may assume that the m_1-chart Γ_1 has an outermost 1-valent vertex with sign -1 and label $m_1 - 1$ and that the m_2-chart Γ_2 has an outermost 1-valent vertex with sign $+1$ and label 1. Let Γ be the composite chart $\Gamma_1\#\Gamma_2 = \iota_0^{m_2-1}(\Gamma_1) \cdot \iota_{m_1-1}^{0}(\Gamma_2)$. This is an $(m_1 + m_2 - 1)$-chart presenting the connected sum $K_1\#K_2$. Put $m = m_1$ and $\ell = m_1 + m_2 - 1$. The chart Γ is shown as in the top left of Figure 14, where Λ_1 is a subchart whose edges are labeled by integers in $\{1, \dots, m-1\}$ and Λ_2 is a subchart whose edges are labeled by integers in $\{m, m+1, \dots, \ell-1\}$. Transform the chart Γ as in Figures 14 and 15: (Step 1) It is a CIII-move; (Step 2) It is a channel change move which is a CI-move; (Step 3) Do the replacement established in Lemma 13; (Step 4) It is a channel change move; (Step 5) It is a CI-move as in Figure 5; (Step 6) It is a combination of C-moves; (Step 7) It is an MI-move with channel change moves; (Step 8) It is the replacement established in Lemma 14; (Step 9) It is an MI-move with channel change moves. Note that after this step, the chart has a free edge of label $\ell - 1$ and the other edges of the chart are labeled by integers smaller than $\ell - 1$. (Step 10) Apply the inverse operation of an MII-move and obtain the $(\ell - 1)$-chart in the figure. (Step 11) It is an MI-move. Thus the $(\ell - 1)$-chart illustrated in the right bottom of Figure 15 is a chart presentation of $K_1\#K_2$. □

6. Examples

Let n be an odd integer with $n \geq 3$ and let Γ_n be the 3-chart illustrated in Figure 16 which consists of two free edges with label 1 and n parallel hoops with label 2 oriented in the same direction surrounding one of the free edges. We denote by K_n the 2-knot presented by the chart Γ_n (which was denoted by $S_{n,1}$ in [16]). Note that K_n is the 2-knot obtained from a $(2, n)$-torus knot by Artin's spinning (cf. [8, 10, 16, 19]).

Lemma 15. $\mathrm{Braid}(K_n) = 3$ *for any odd integer* $n \geq 3$. *If* $n \neq n'$, *then* K_n *is not equivalent to* $K_{n'}$. *Hence, there exist an infinite series of* 2*-knots of braid index* 3.

Proof. The first Alexander module, the $\Lambda = \mathbb{Z}[t, t^{-1}]$-module of the homology $H_1(\widetilde{E}; \mathbb{Z})$ of the universal abelian covering $\widetilde{E} \to E$ of the knot exterior E of the 2-knot K_n is isomorphic to that of a $(2, n)$-torus knot. Thus, we see that K_n is non-trivial and that K_n is not equivalent to $K_{n'}$ if $n \neq n'$. Since any non-trivial 2-knot has braid index greater than 2, we have the result. □

Lemma 16. $\mathrm{Braid}(K_n\#K_n) = 4$ *for any odd integer* $n \geq 3$.

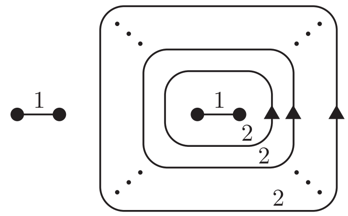

FIGURE 16.

Proof. By Theorem 3 and Lemma 15, we have $\mathrm{Braid}(K_n \# K_n) \leq 4$. Using the standard form of a 3-braid 2-knot [16], it is seen that any 3-braid 2-knot is a 1-fusion ribbon 2-knot, that is, it is obtained from a pair of trivial 2-spheres in $\mathbb{R}^4$ by surgery along a single 1-handle. So its first Alexander module must be cyclic. On the other hand, the first Alexander module of $K_n \# K_n$ is the direct sum of two copies of $\Lambda/(f_n)$, which is not cyclic, where f_n is the Alexander polynomial of a $(2,n)$-torus knot. Hence, we have $\mathrm{Braid}(K_n \# K_n) > 3$. □

Proof of Corollary 4. Consider the family $\{K_n \# K_n\}$ for odd integers $n \geq 3$. If $n \neq n'$, then $K_n \# K_n$ and $K_{n'} \# K_{n'}$ are not equivalent, since their first Alexander modules are not isomorphic. In particular, $K_3 \# K_3$ is the 2-knot obtained from a granny knot by Artin's spinning. □

Questions. Do there exist 2-knots K_1 and K_2 with

$$\mathrm{Braid}(K \# K') < \mathrm{Braid}(K_1) + \mathrm{Braid}(K_2) - 2?$$

In particular, do there exist 2-knots K_1 and K_2 with

$$\mathrm{Braid}(K_1 \# K_2) = \mathrm{Braid}(K_1) = \mathrm{Braid}(K_2) = 3?$$

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Department of Mathematics, Hiroshima University, Higashi-Hiroshima, 739-8526, Japan

E-mail address: kamada@math.sci.hiroshima-u.ac.jp

Department of Mathematics, Chiba University, Inage, Chiba, 263-8522, Japan

E-mail address: satoh@math.s.chiba-u.ac.jp

Japan Tokushima Prefectural, Mental Health & Welfare Center, 3-80 Shinkura, Tokushima, 770-0855, Japan

E-mail address: manabu12@khaki.plala.or.jp